Exploring superfluidity in flat-band Bose systems through quantum geometry

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17th September — Superfluctuations2024 Salerno

The quantum geometric tensor

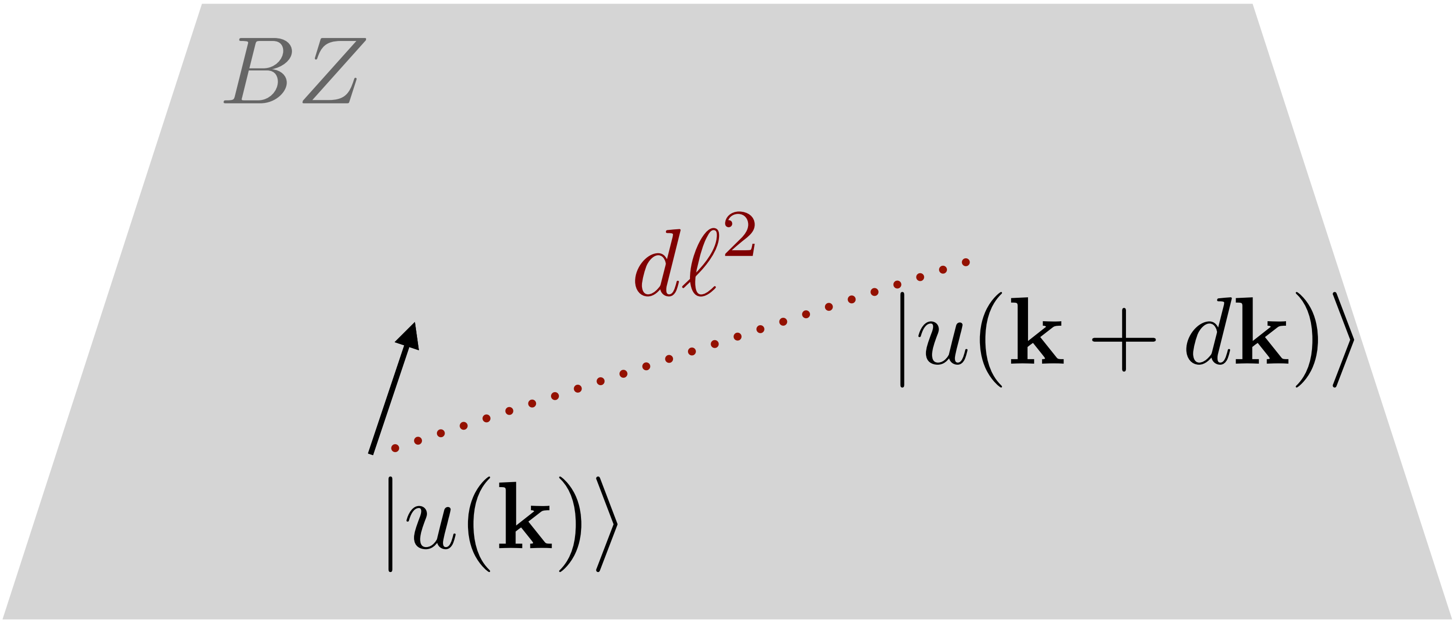
Variation between quantum states

Metric for the distance between quantum states

Provost & Vallee, *Comm. Math. Phys.* **76**, 289 (1980)

$$d\ell^2 = ||u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})||^2 = \langle u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})|u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})\rangle$$

$$\approx \sum_{i,j} \langle \partial_{k_i} u | \partial_{k_j} u \rangle dk_i dk_j$$



Quantum geometric tensor is defined

$$T_{ij} = \langle \partial_{k_i} u | (1 - |u\rangle\langle u|) | \partial_{k_j} u \rangle$$

$$g_{ij} = \mathrm{Re}\{T_{ij}\}$$

Fubini-Study metric (quantum metric)

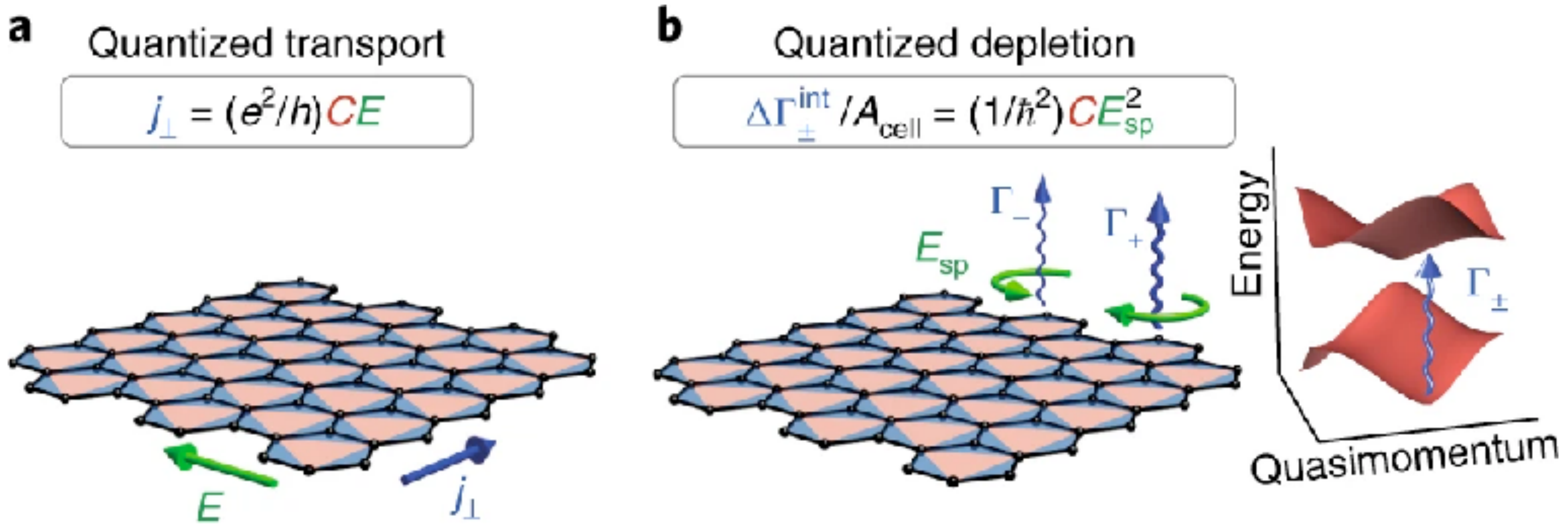
$$B_{ij} = -\mathrm{Im}\{T_{ij}\}$$

Berry Curvature

Quantum geometric tensor observations

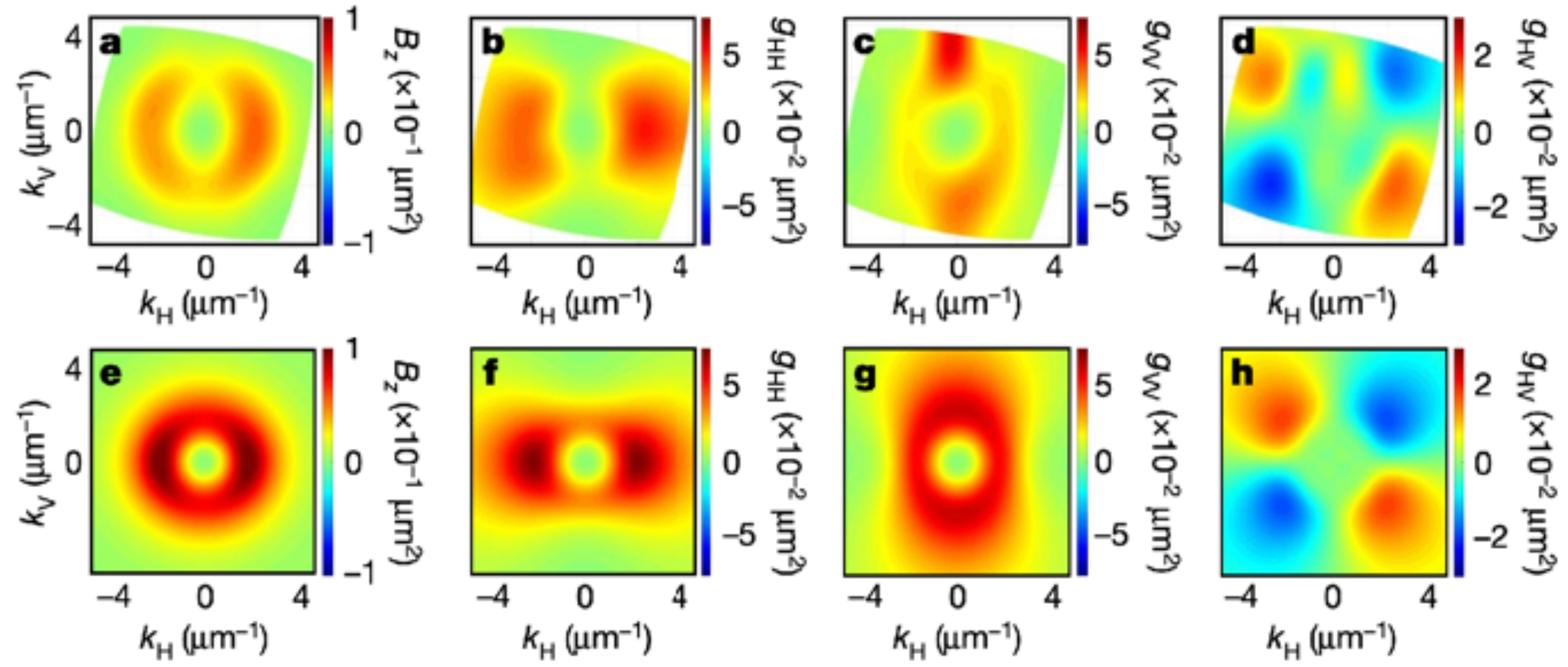
Not an abstract quantity

Ultracold atoms



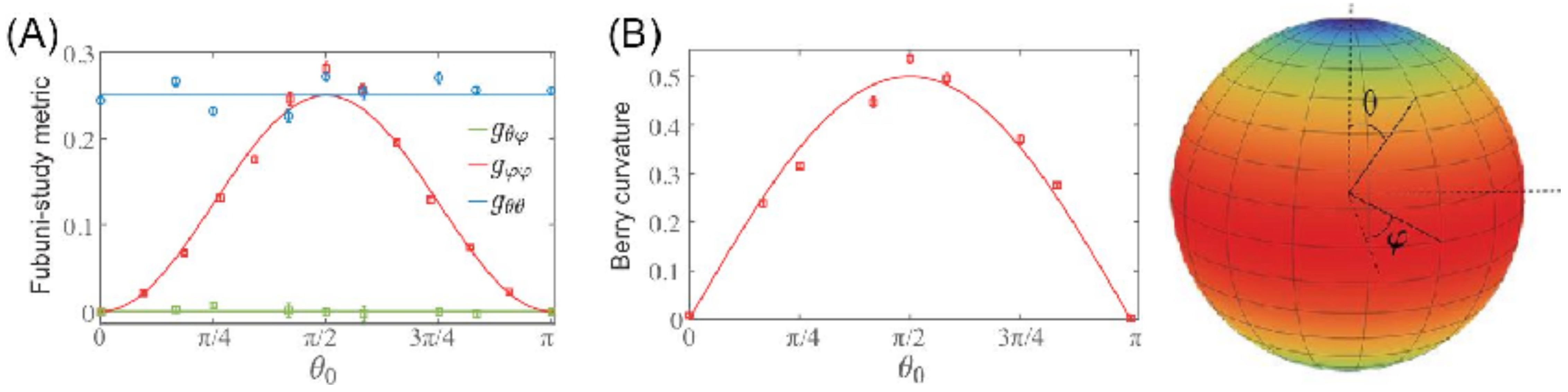
Asteria et al., *Nat. Phys.* **15**, 449 (2019)

Microcavity exciton-polariton



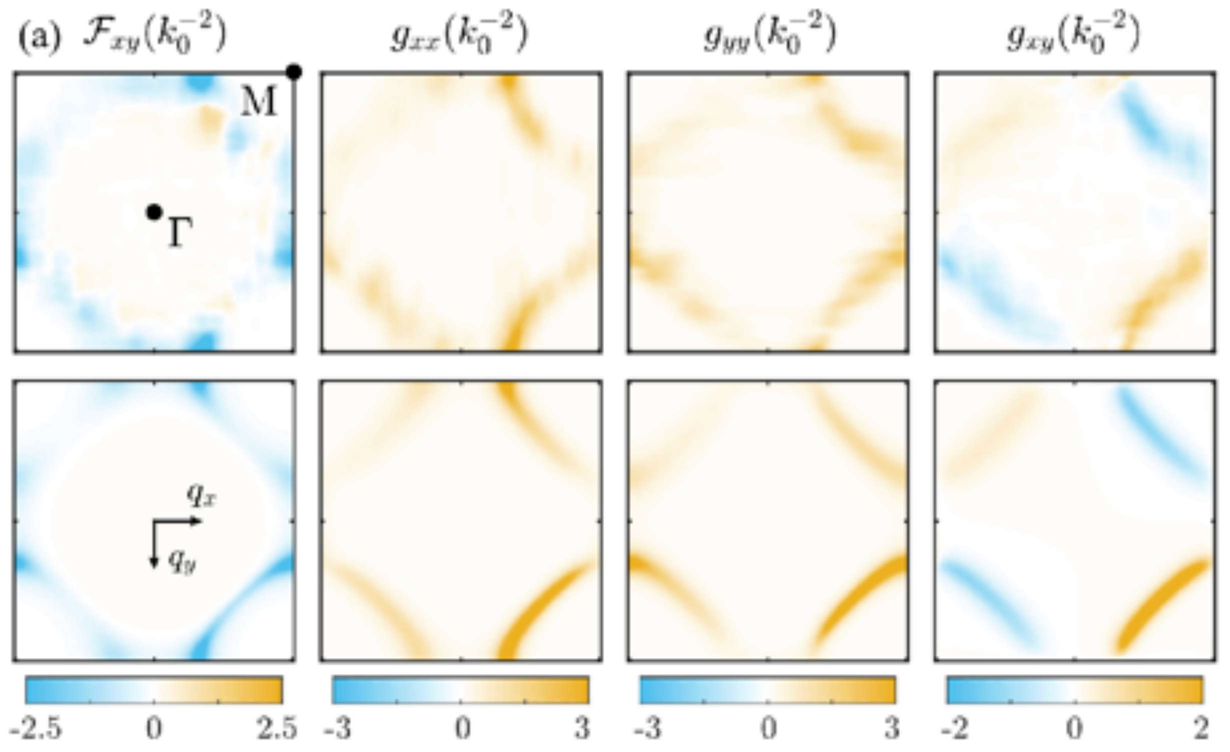
Gianfrate et al., *Nature* **578**, 381 (2020)

Coupled qubits in diamond



Min et al. *Natl. Sci. Rev.* **7**, 254 (2020)

Optical Raman lattice



Chang-Rui et al., *Phys. Rev. Research* **5**, L032016 (2023)

The quantum geometric tensor in Physics

<u>Real part: Fubini-Study Metric</u>

- Signatures in current noise
  Neupert, Chamod & Mudry *Phys. Rev. B* (2013)
- Fractional Chern insulators
  Dobardžić et al., *PRB* (2013) — Roy, *Phys. Rev. B* (2014)
- Superfluidity & Superconductivity in flat bands
  Peotta & Törmä, *Nat. Commun.* (2015)
  Törmä, Peotta & Bernevig, *Nat. Rev. Phys.* (2022)
- Orbital magnetic susceptibility
  Piéchon et al., *Phys. Rev B* (2016)
- Excitonic Lamb shift
  Srivastava & Imamoğlu, *Phys. Rev. Lett.* (2015)
- Topological semimetals characterization
  Salerno, Goldman & Palumbo, *Phys. Rev. Research* (2020)
- Bose-Einstein condensation in flat bands
  Julku, Bruun & Törmä, *Phys. Rev. Lett.* (2021)
  Julku, Salerno & Törmä, *Low Temp. Phys.* (2023)
- Drude weight in flat band many-body systems
  Salerno, Ozawa & Törmä, *Phys. Rev. B* (2023)

<u>Imaginary part: Berry Curvature</u>

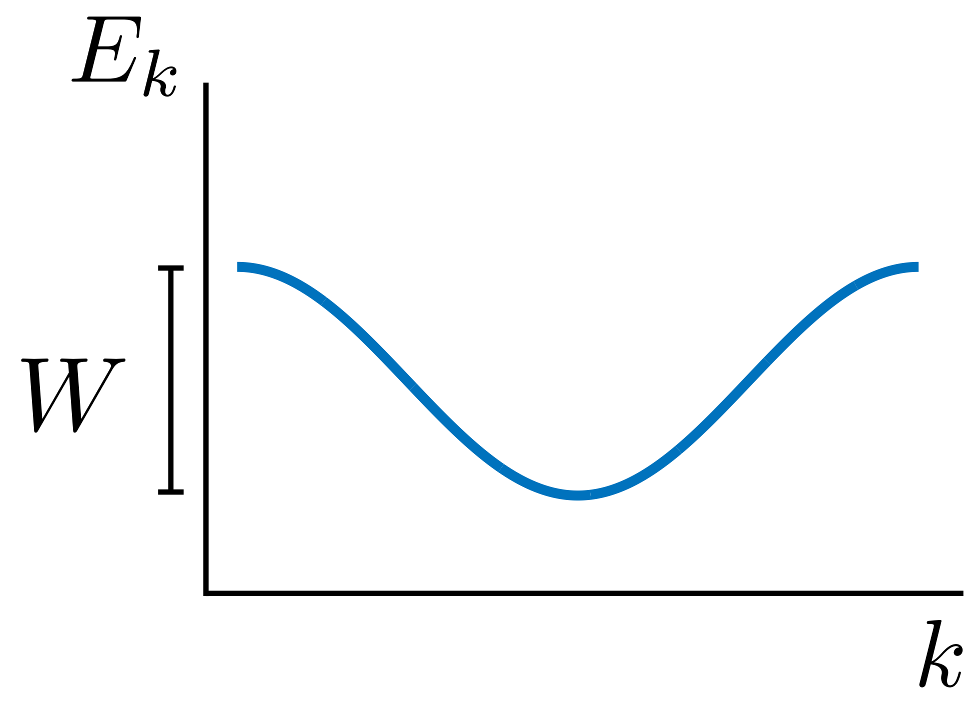
- Quantum Hall Effects (Integer, Anomalous, Fractional, Valley, Spin, 4D, …)
- Steady-state Hall response and circular dichroism
- Semiclassical dynamics
- Topological pumping
- Topological semimetals
- Topological defects and textures
- Topological Superconductors

A not complete list!

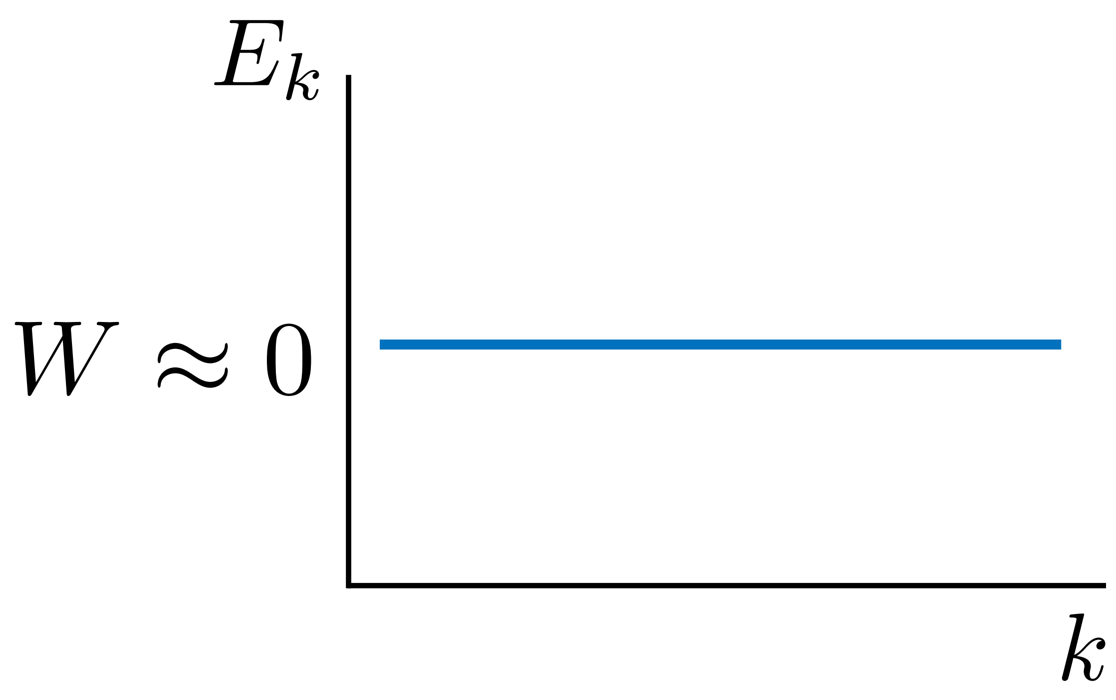
Quantum geometry and flat bands

Flat bands

Where interactions dominate



$U \ll W$



$U \gg W$

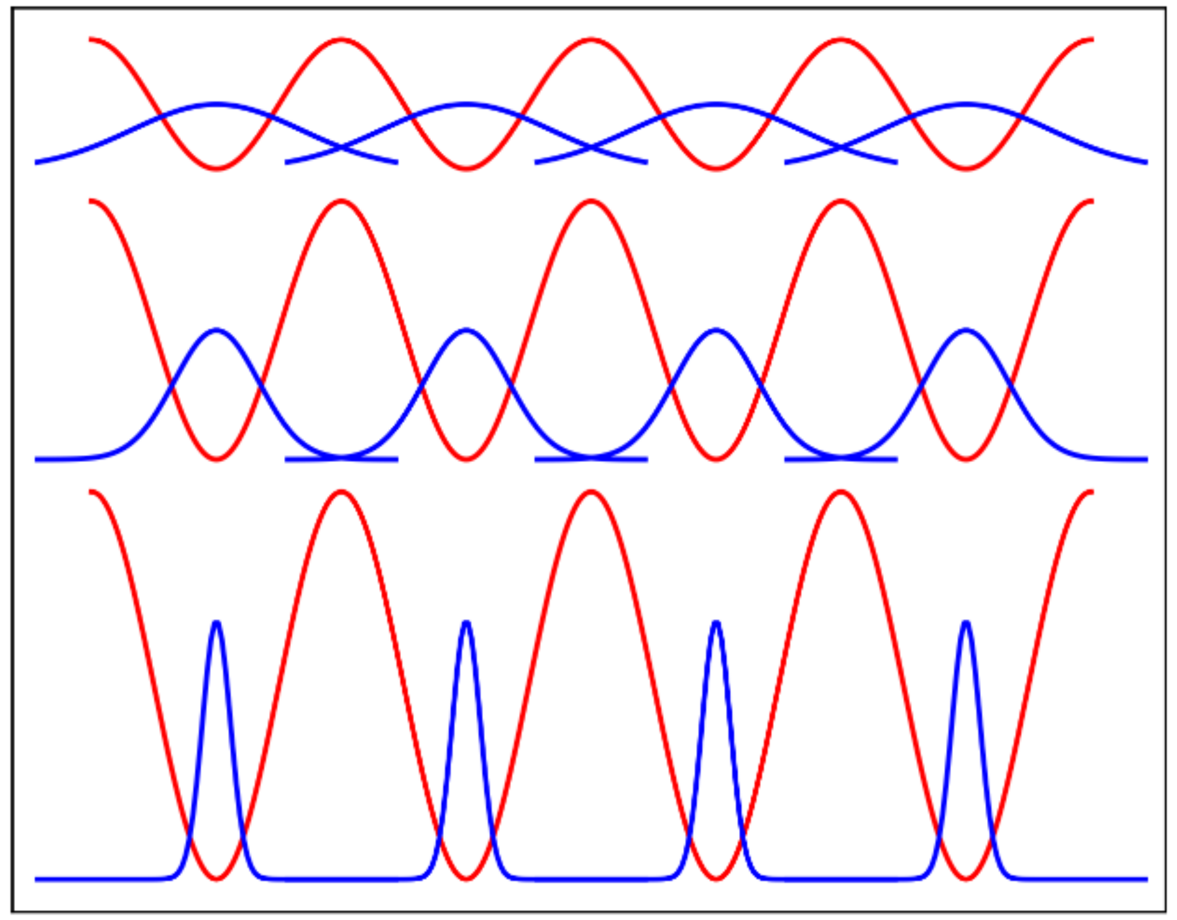
Efficient charge transport set by the group velocity. Minimal sensitivity to weak interactions.

Any perturbation, no matter how weak, can set a new dominant energy scale for the flat band states and qualitatively change their transport properties.

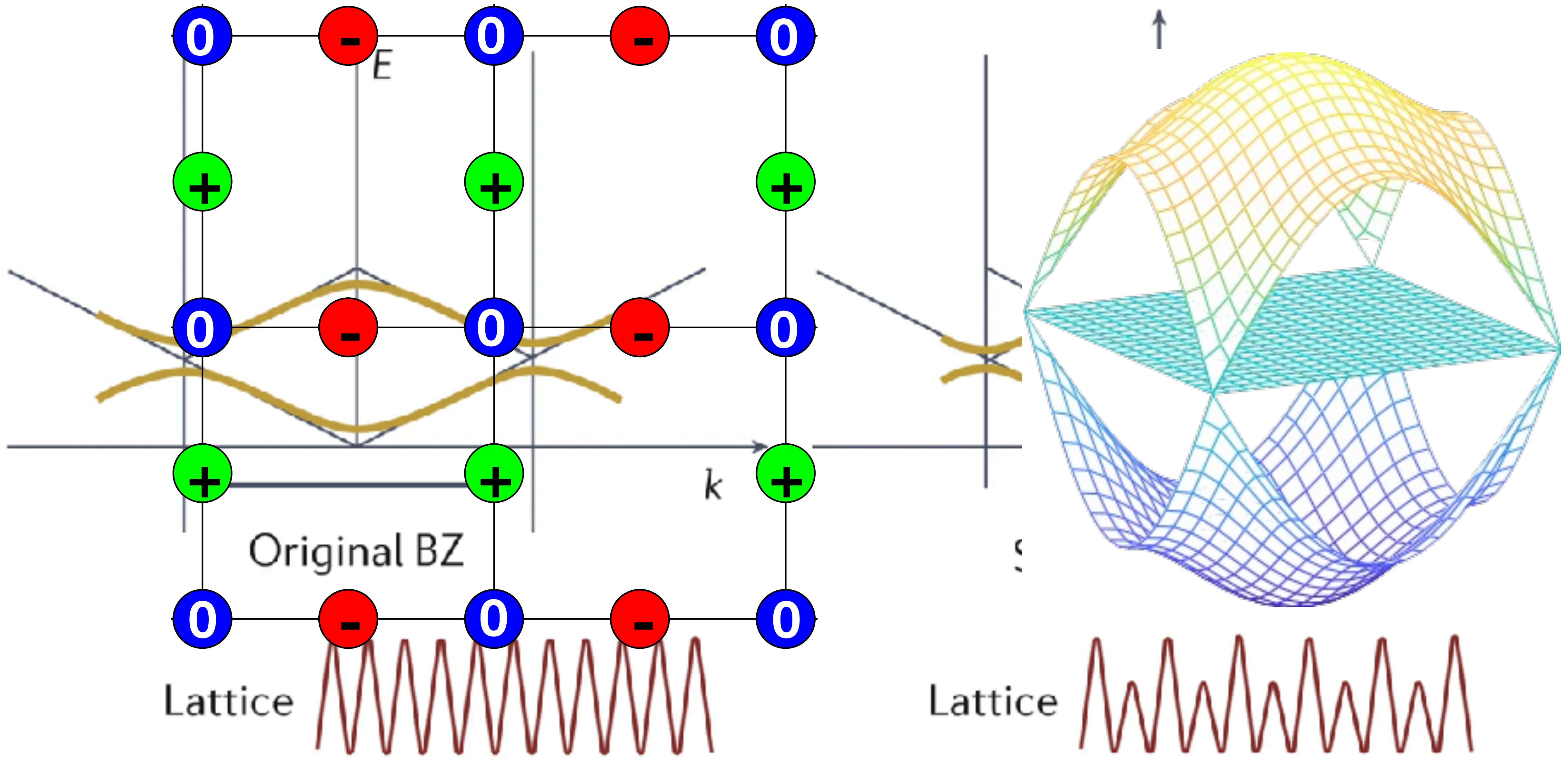
Flat bands

The origin of localization

Trivial case: the extreme atomic limit



Non-trivial: destructive interference effects in tunneling
Non-trivial: band flattening with superlattices (TBG)



Review: Leykam, Andreanov & Flach, *Adv. Phys.: X* **3**, 1473052 (2018)

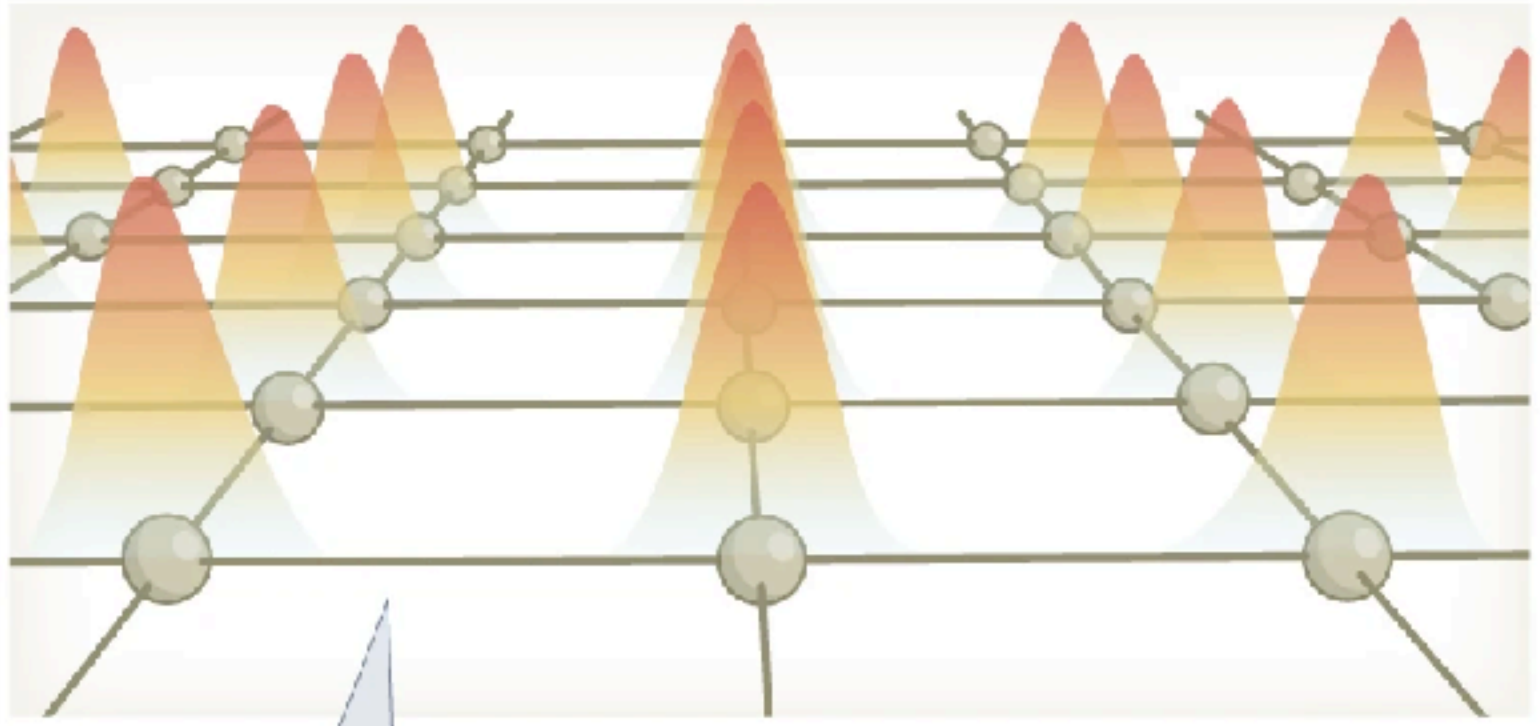
Cold atoms realizations:
Lieb (Takahashi, Bakr); *Kagome* (Sengstock, Stamper-Kurn, Schneider, Bloch), *Haldane/Harper* (Esslinger, Bloch, Ketterle, Sengstock, Weitenberg, Aidelsburger)

Flat bands

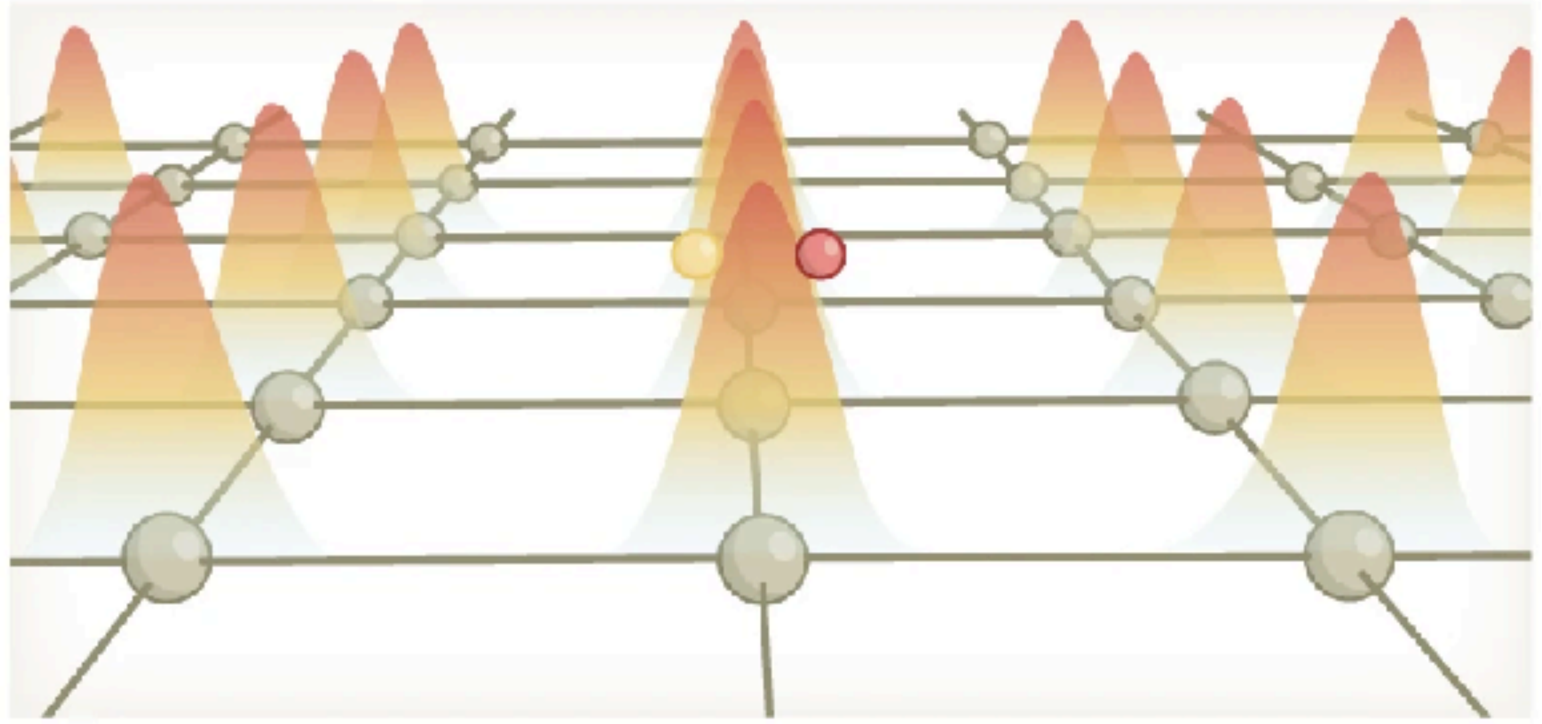
Wannier functions and interactions

Trivial case: the extreme atomic limit

$U = 0$

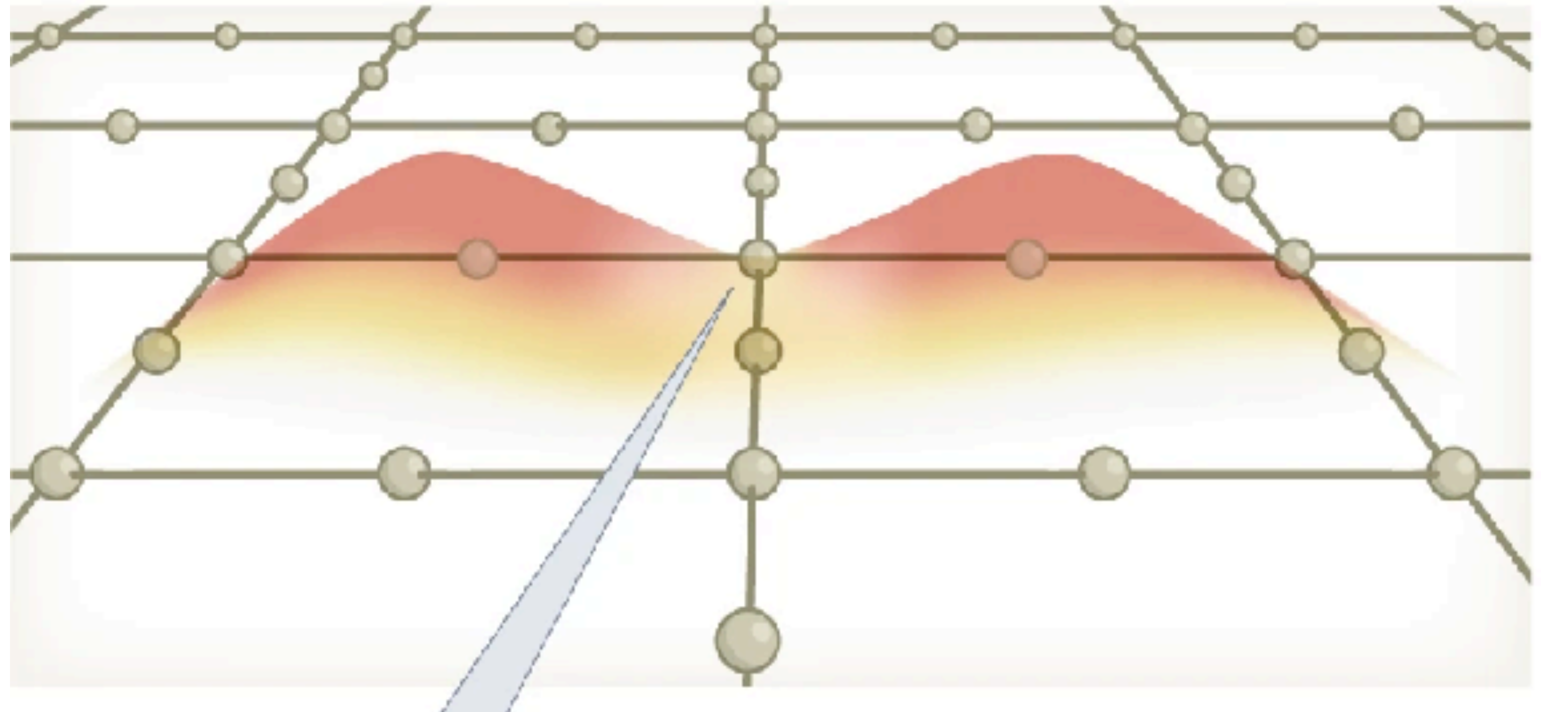


$U \neq 0$

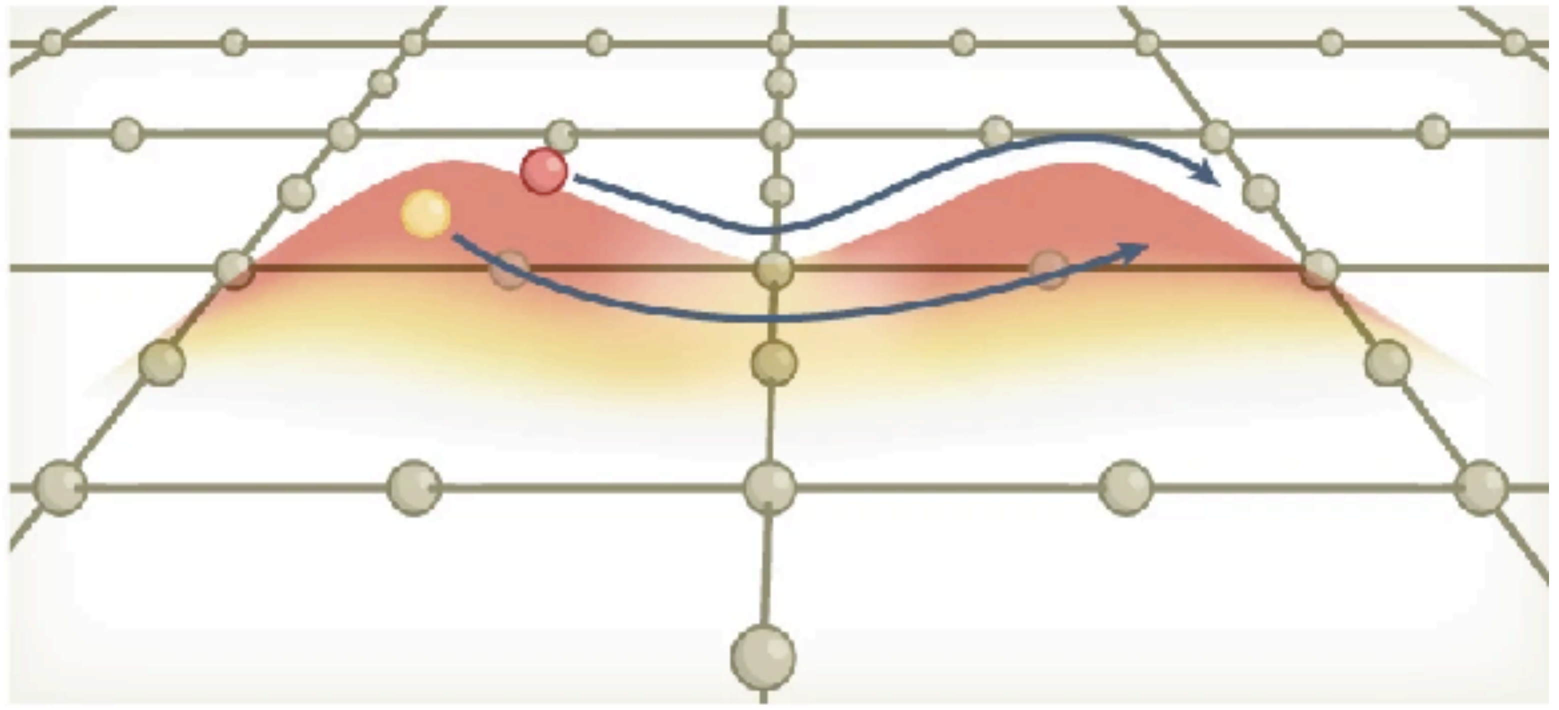


Non-trivial: destructive interference

$U = 0$



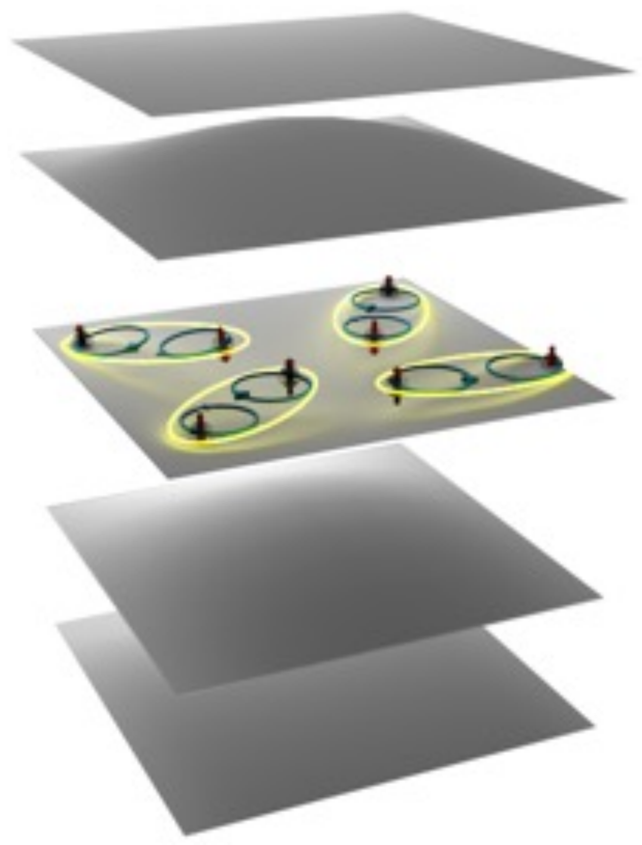
$U \neq 0$



Törmä, Peotta & Bernevig, *Nat. Rev. Phys.* **4**, 528 (2022)

Superfluid weight in a flat band system

The geometrical contribution



$$D_s = D_{s,\mathrm{conventional}} + D_{s,\mathrm{geometrical}}$$

$$\propto \partial_{k_i}\partial_{k_j}E_{\mathbf{k}}$$

It is zero in a flat band

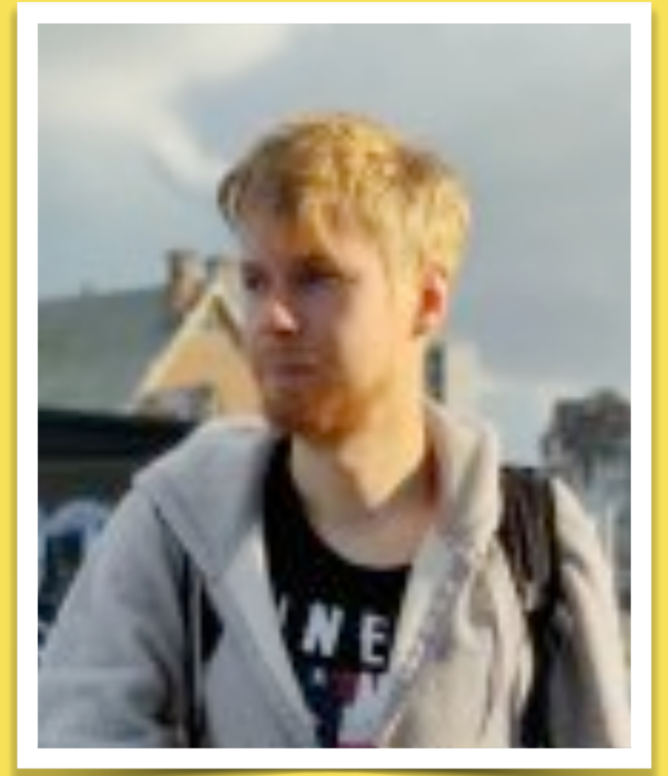
$$\propto Ug_{ij} \geq UC$$

Can be nonzero also in a flat band and it is present only in a multi band case. Proportional to the quantum metric and lower bounded by the Chern number.

Peotta & Törmä, *Nat. Comm.* **6**, 8944 (2015), Liang et al., *Phys. Rev. B* **95**, 024515 (2017), Huhtinen et al., *Phys. Rev. B* **106**, 014518 (2022)

Superfluidity of flat band BECs

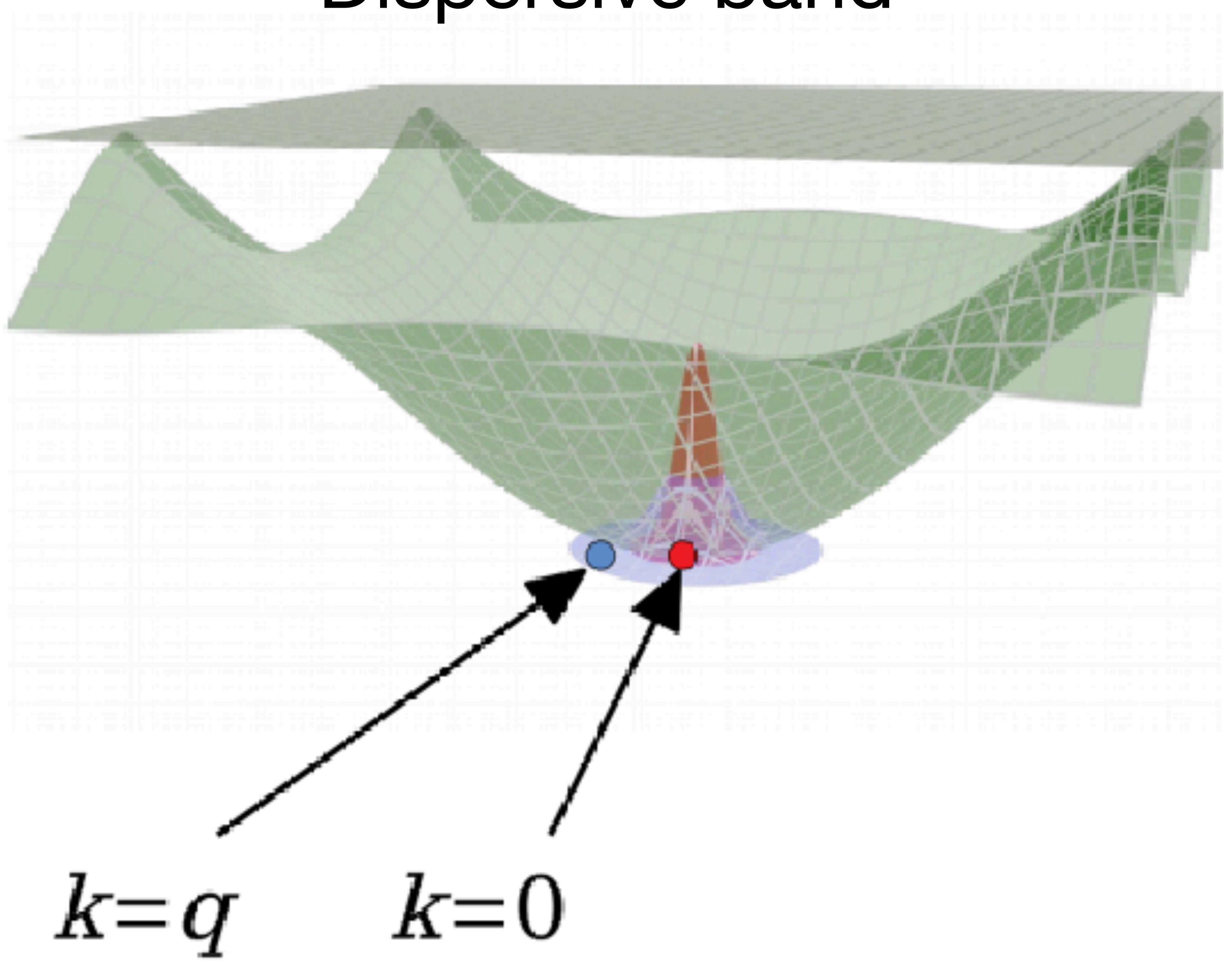
Julku, <u>Salerno</u> & Törmä, *Low Temp. Phys.* **40**, 701 (2023)



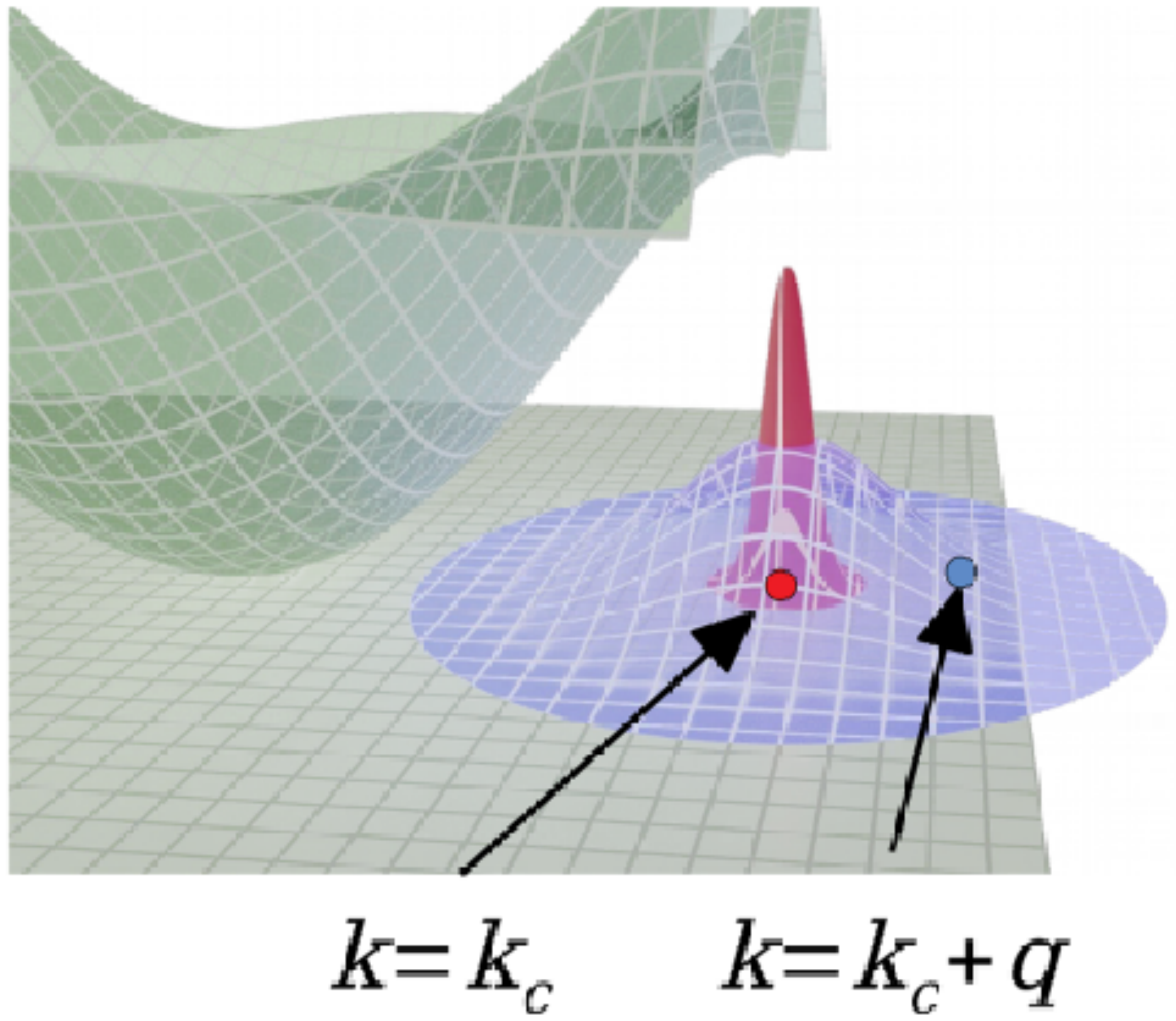
Bose-Einstein condensation in flat bands

Is it even possible?

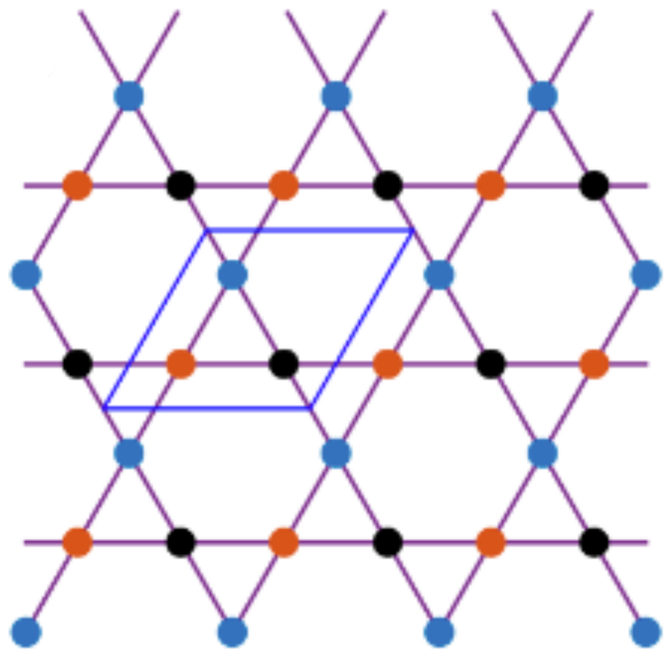
Dispersive band



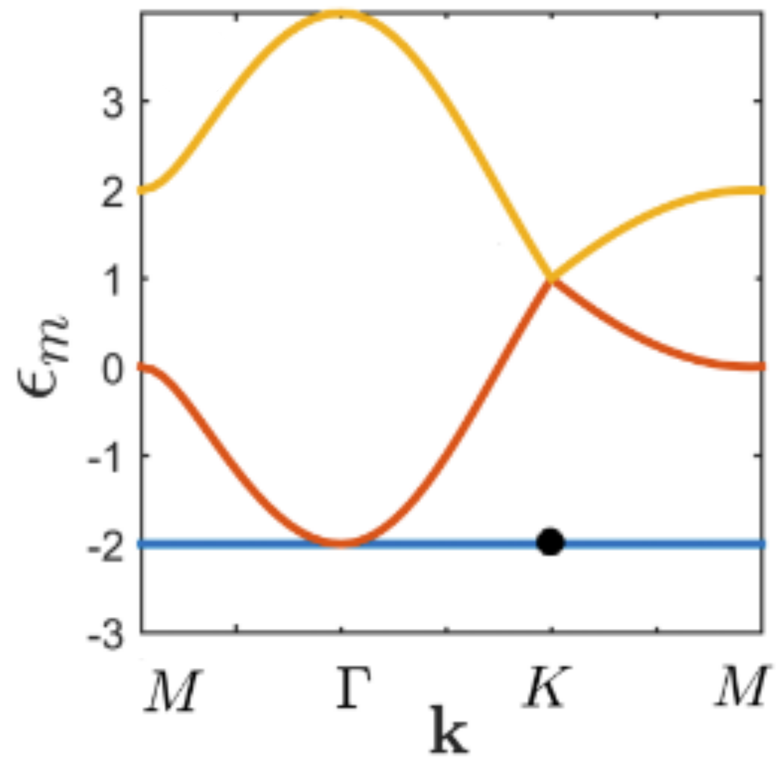
Flat band



Kagome lattice



Condensation point



BEC forms where mean-field interactions are minimized — particle density is uniform among the sublattices.

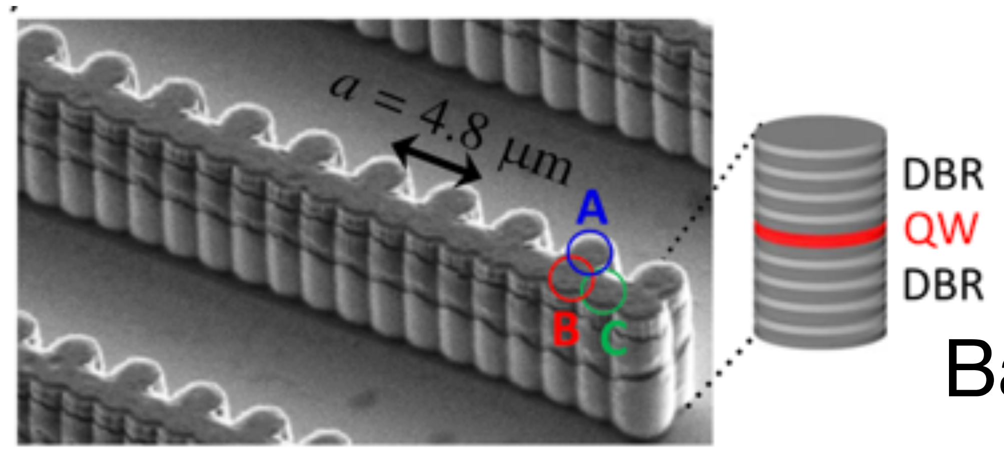
Huber & Altman, *Phys. Rev. B* **82**, 184502 (2010)

You et al., *Phys. Rev. Lett.* **109**, 265302 (2012)

Julku, Bruun & Törmä, *Phys. Rev. B* **104**, 144507 (2021)

Julku, Salerno & Törmä, *Low Temp. Phys.* **40**, 701 (2023)

Sensitive to onsite disorder in experiments: fragmented BECs.



Baboux et al., *Phys. Rev. Lett.* **116**, 066402 (2016)

Bose-Einstein condensation in flat bands

The effect of condensate quantum distance on stability

At U→ 0, excitations do not cost energy. Can a BEC be stable?

The excitation density within Bogoliubov theory yields

$$\lim_{U\to 0} n_{\mathrm{ex}}(\mathbf{k}) = \lim_{U\to 0} \langle \delta c^\dagger_{\mathbf{k}} \delta c_{\mathbf{k}} \rangle = \frac{1 - D(\mathbf{q})}{2D(\mathbf{q})}$$

$$c_{\mathbf{k},\alpha} = \sqrt{N n_0} \langle \alpha | \phi_0 \rangle \delta_{\mathbf{k},\mathbf{k}_c} + \delta c_{\mathbf{k},\alpha}$$

n_0 Condensation density per unit cell

$|\phi_0\rangle \equiv |u(\mathbf{k}_c)\rangle$ Condensate state

α Sublattice degree of freedom

Condensate quantum distance involves overlaps of the condensate state with neighboring Bloch states

$$D(\mathbf{q}) \equiv \sqrt{1 - |d(\mathbf{q})|^2}$$

$$d(\mathbf{q}) \equiv M \sum_{\alpha} \langle u_1(\mathbf{k_c} + \mathbf{q}) | \alpha \rangle \langle \alpha | \phi_0 \rangle \langle \phi_0^* | \alpha \rangle \langle \alpha | u_1^*(\mathbf{k_c} - \mathbf{q}) \rangle$$

If flat band states overlapping perfectly D(**q**) = 0, BEC is not stable as excited fraction diverges.
Finite quantum distance between Bloch states sets the limit for excitation density, **determining if BEC is stable.**

Julku, Bruun & Törmä, *Phys. Rev. Lett.* **127**, 170404 (2021)

Julku, Salerno & Törmä, *Low Temp. Phys.* **40**, 701 (2023)

Bose-Einstein condensation in flat bands

Quantum distance and quantum metric

Condensate quantum distance involves overlaps of the condensate state with neighboring Bloch states

$$D(\mathbf{q}) \equiv \sqrt{1 - |d(\mathbf{q})|^2} \qquad d(\mathbf{q}) \equiv M \sum_{\alpha} \langle u_1(\mathbf{k_c} + \mathbf{q})|\alpha\rangle\langle\alpha|\phi_0\rangle\langle\phi_0^*|\alpha\rangle\langle\alpha|u_1^*(\mathbf{k_c} - \mathbf{q})\rangle$$

If $|u^*(\mathbf{k})\rangle = |u(\mathbf{k})\rangle$ condensate quantum distance gets to the quantum metric $D(\mathbf{q}) \to 4\sum_{\mu\nu} q_\mu q_\nu g_{\mu\nu}(\mathbf{k}_c)$

otherwise

$$\tilde{g}_{\mu\nu}(\mathbf{k}_c) = \frac{1}{2}\left[g_{\mu\nu}(\mathbf{k}_c) + \mathrm{Re}\left(\langle\partial_\mu\phi_0|\phi_0\rangle\langle\phi_0|\partial_\nu\phi_0\rangle\right) + 2M\mathrm{Re}\left(\sum_\alpha \langle\partial_\mu\phi_0|\alpha\rangle\langle\alpha|\phi_0\rangle\langle\phi_0^*|\alpha\rangle\langle\alpha|\partial_\nu\phi_0^*\rangle\right)\right]$$

This generalized versions of the quantum metric and the quantum distance are *independent on orbital positions*. Only in highly symmetric lattices, BEC properties are connected to the usual quantum metric and the Hilbert-Schmidt quantum distance.

Julku, Bruun & Törmä*, Phys. Rev. Lett.* **127**, 170404 (2021)
Julku, Salerno & Törmä*, Low Temp. Phys.* **40**, 701 (2023)

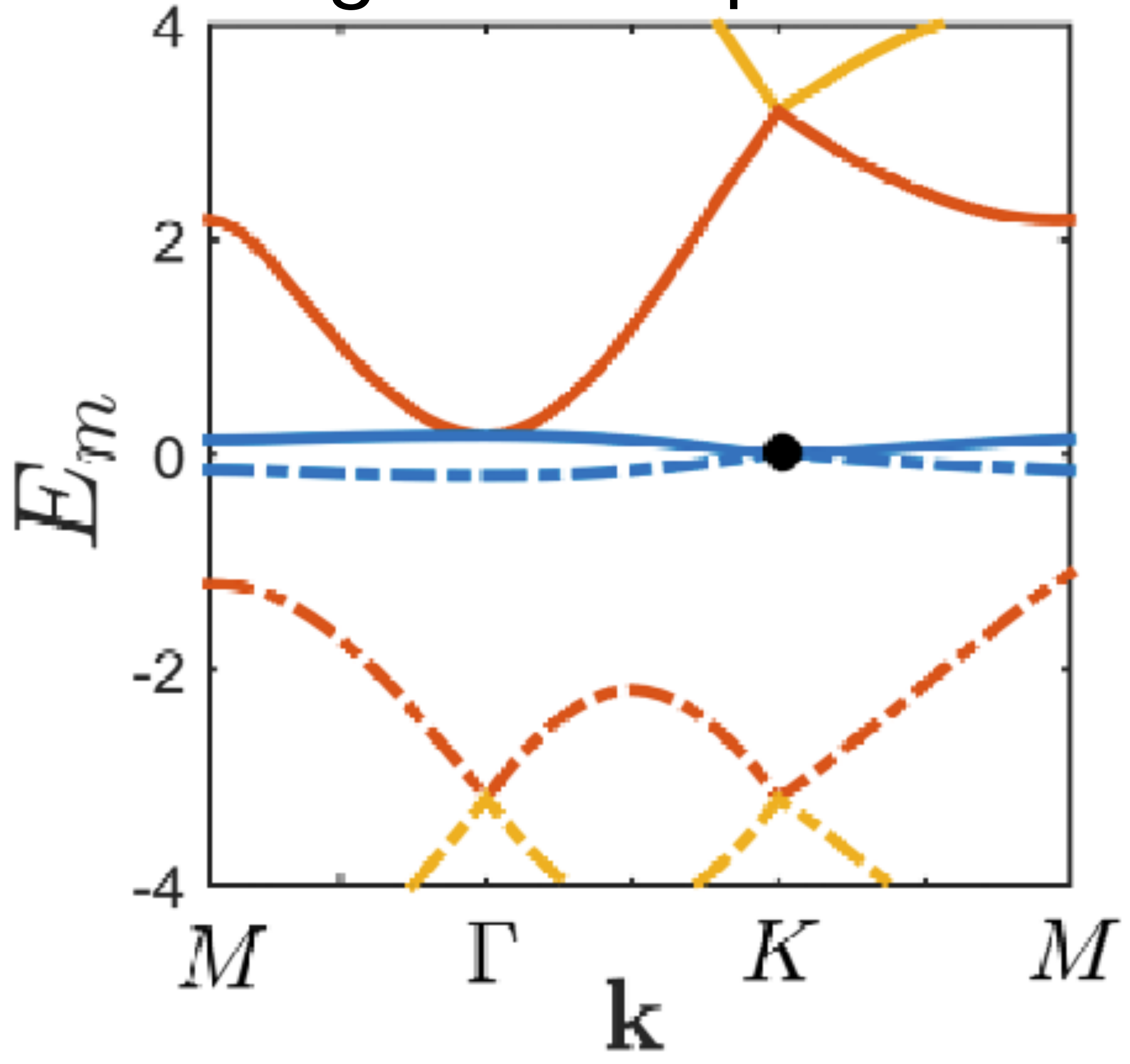
See also: Iskin, *Phys. Rev. A* **107**, 023313 (2023)

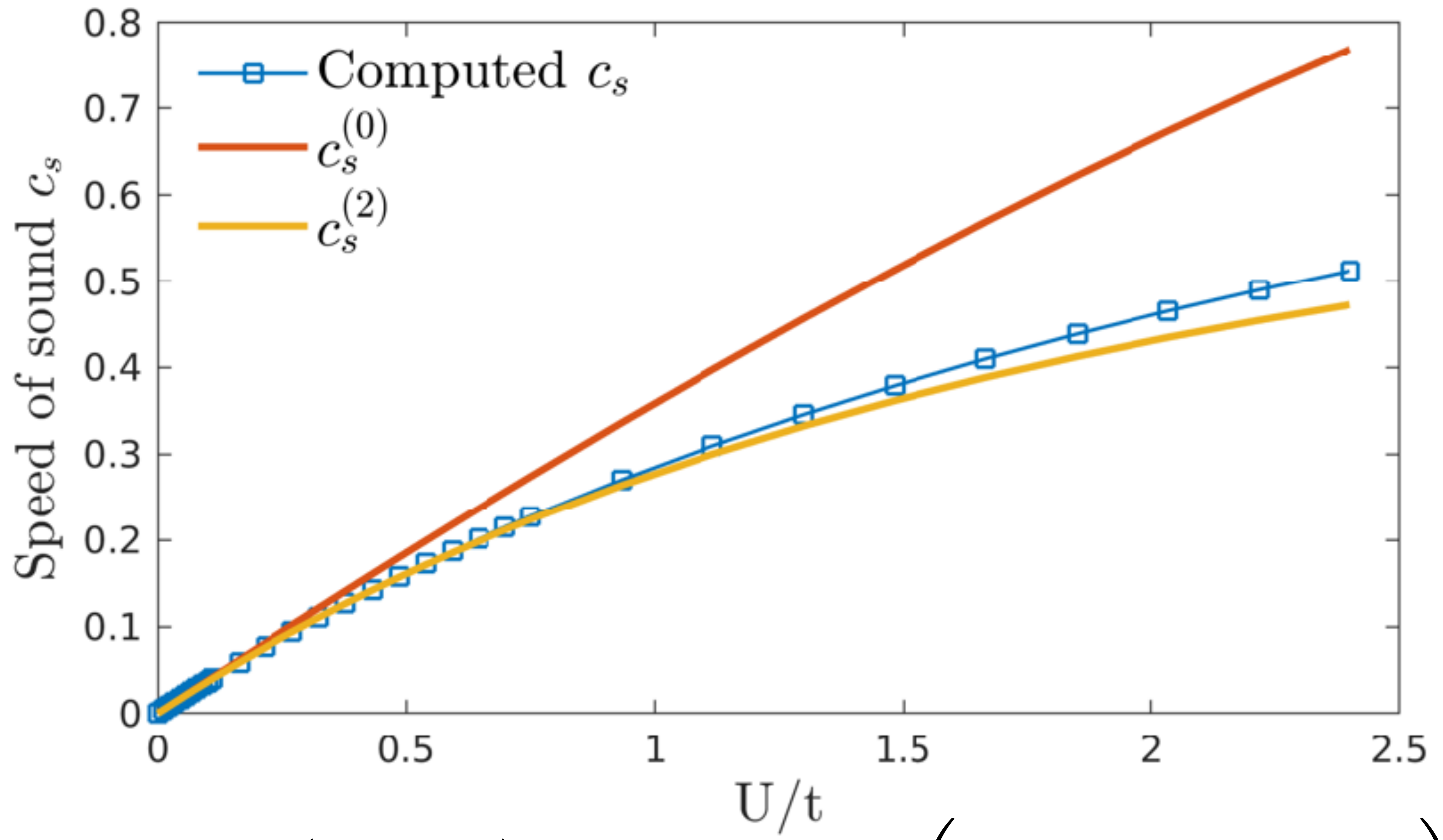
Bose-Einstein condensation in flat bands

The effect of condensate quantum distance on sound velocity

Condensate quantum distance also dictates the speed of sound

Bogoliubov spectrum





$$c_s = \lim_{\mathbf{q}\to 0} \frac{E(\mathbf{k}_c + \mathbf{q})}{|\mathbf{q}|} \propto \frac{U n_0}{M} D(\mathbf{q}) \left(1 - \frac{1}{1 + \Delta_g / \frac{U n_0}{M}}\right)$$

Results for Kagome lattice

Julku, Salerno & Törmä, *Low Temp. Phys.* **40**, 701 (2023)

Bose-Einstein condensation in flat bands

The effect of condensate quantum distance on correlations

Condensate quantum distance also related to fluctuations in second-order coherence function

$$g^{(2)} = \frac{\sum_i \langle n_i n_i \rangle - n_{\mathrm{tot}}}{N n_{\mathrm{tot}}^2} = g_1^{(2)} + g_2^{(2)} \boxed{+ g_3^{(2)}}$$

$g_1^{(2)}$ Mean field contribution arising from the BEC.

$g_2^{(2)}$ Contribution from BEC and fluctuations, involving the overlap between Bogoliubov excitation and the condensate state.

$g_3^{(2)}$ Contribution from quantum fluctuations of Bogoliubov excitations. This term is governed by the quantum metric for vanishing interactions.

$$\lim_{U \to 0} g_3^{(2)} = \frac{1}{8 N^2 n_{\mathrm{tot}}^2} \times \sum_{k, k' \neq k_c} \left(\frac{d(\mathbf{k}) d^*(\mathbf{k}')}{D(\mathbf{k}) D(\mathbf{k}')} + \frac{(1 - D(\mathbf{k}))(1 + D(\mathbf{k}'))}{D(\mathbf{k}) D(\mathbf{k}')} \right)$$

Negligible for a dispersive band BEC. Guaranteed to be nonzero by the quantum distance D(**k**) in a flat band BEC!

Julku, Bruun & Törmä, Phys. Rev. B 104, 144507 (2021)

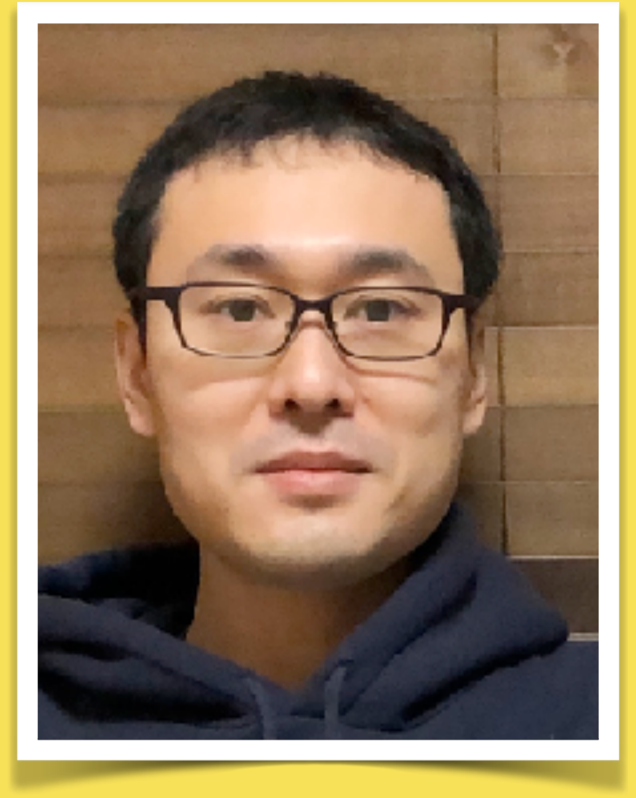
Quantum metric in Bose systems

Is this the end?

- The quantum metric defined dictates the speed of sound and the excitation fraction of flat band BECs within Bogoliubov theory.
- So far we have considered only single-particle (Bloch states) quantum metric and mean-field approaches.
- What about a *many-body quantum metric* and exact approaches?

The many-body quantum metric and the Drude weight

Salerno, Ozawa & Törmä, *Phys. Rev. B* **108**, L140503 (2023)



The many-body quantum metric

Defined on many-body states with respect to the twisted boundary condition phase

$$\mathfrak{g}(\phi) = \mathrm{Re}\left[\langle \partial_\phi \Psi_0 | \left(1 - |\Psi_0\rangle\langle\Psi_0|\right) |\partial_\phi \Psi_0\rangle\right]$$

determines the “quantum distance” along a given path in a parameter space.

Many-body generalization of the quantum metric:

$$\boxed{\mathfrak{g}(0) = \mathrm{Re}\left[\sum_{m\neq 0} \frac{|\langle \Psi_m | \partial_\phi \hat{H}(\phi) | \Psi_0\rangle|^2}{(E_m(0) - E_0(0))^2}\right]}$$

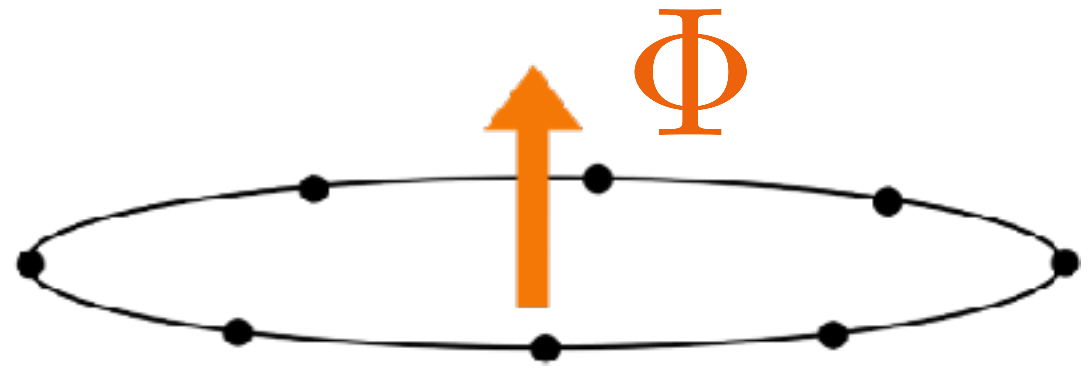
Ozawa & Goldman, *Phys. Rev. Research* **1**, 032019 (2019)

Souza, Wilkens & Martin, *Phys. Rev. B* **62**, 1666 (2000)

Drude weight and many-body quantum metric

System's response to a small external flux
(twisted boundary conditions)

$$D_w = \pi L \left. \frac{\partial^2 E(\Phi)}{\partial \Phi^2} \right|_{\Phi=0}$$



Perturbation theory yields:

$$D_w = -\frac{\pi}{L}\langle \Psi_0 | \hat{H}_{\mathrm{kin}} | \Psi_0 \rangle - \frac{2\pi}{L} \underbrace{\sum_{m\neq 0} \frac{|\langle \Psi_m | \hat{J} | \Psi_0 \rangle|^2}{E_m(0) - E_0(0)}}_{\geq \mathfrak{g}(0)\cdot\varepsilon}$$

$$\mathfrak{g}(0) = \mathrm{Re}\left[\sum_{m\neq 0} \frac{|\langle \Psi_m | \partial_\phi \hat{H}(\phi) | \Psi_0 \rangle|^2}{(E_m(0) - E_0(0))^2}\right]$$

$$\partial_\phi H(\phi) = \hat{J}$$

Drude weight can be bounded by the **many-body quantum metric**
if the system is gapped and higher bands don't contribute

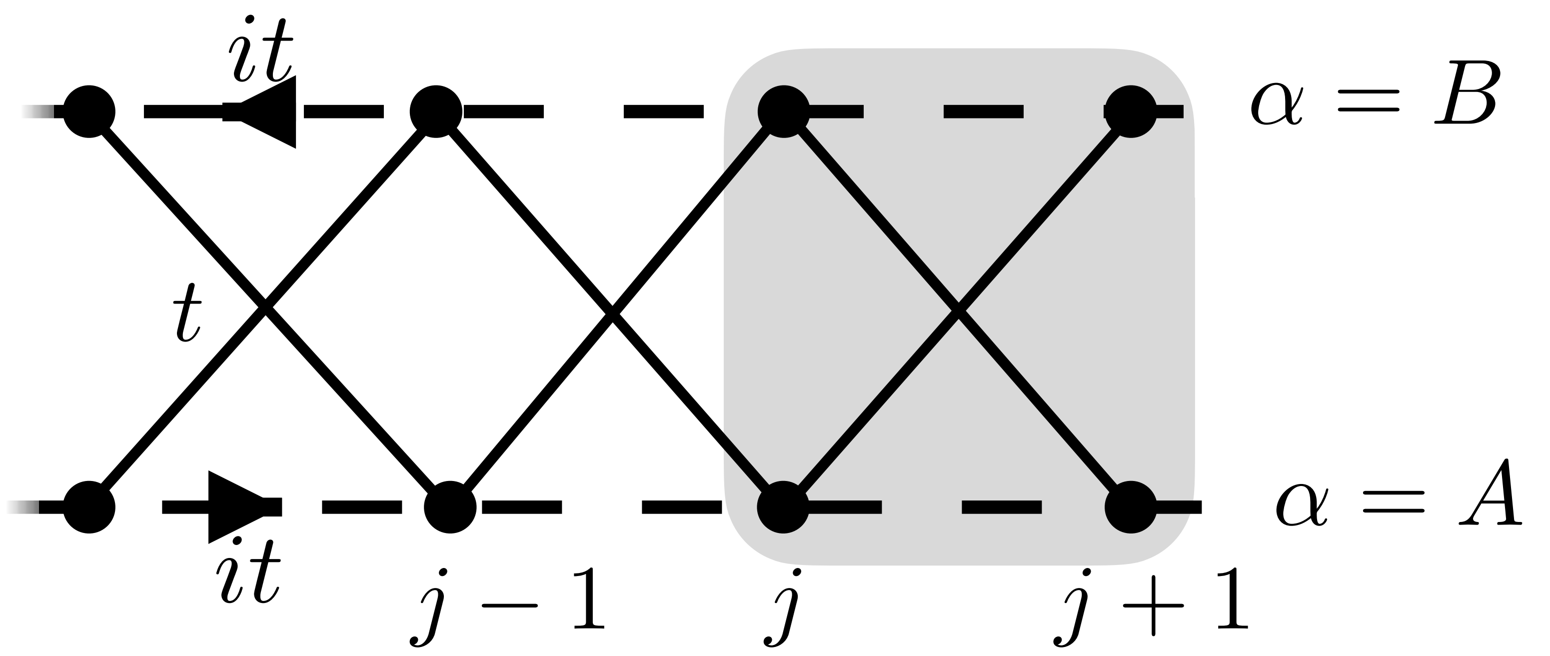
$$D_w \leq -\frac{\pi}{L}\langle \Psi_0 | \hat{H}_{\mathrm{kin}} | \Psi_0 \rangle - \frac{2\pi}{L}\mathfrak{g}(0)\cdot\epsilon$$

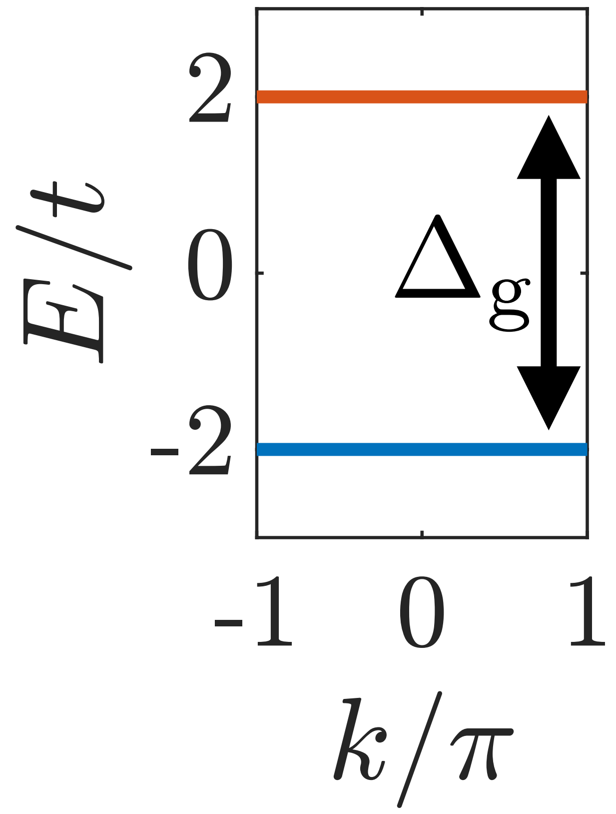
Independent of particle statistics and spatial dimensions!

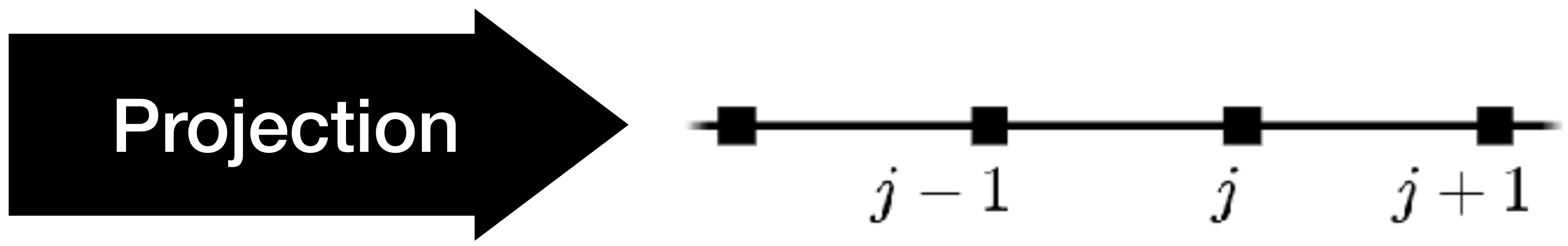
Roth & Burnett, *Phys. Rev. A* **68**, 023604 (2003)
Verma et al., *PNAS* **118**, e2106744118 (2021)

Salerno, Ozawa & Törmä, *Phys. Rev. B* **108**, L140503 (2023)

Bosons in a Creutz ladder with onsite interactions







The lowest flat band state $\hat{W}_j^\dagger|0\rangle = \frac{1}{2}\left(\hat{a}_j^\dagger + i\hat{b}_j^\dagger - ie^{i\Phi/L}\hat{a}_{j+1}^\dagger - e^{i\Phi/L}\hat{b}_{j+1}^\dagger\right)|0\rangle$

The Hamiltonian projected onto the lower flat band

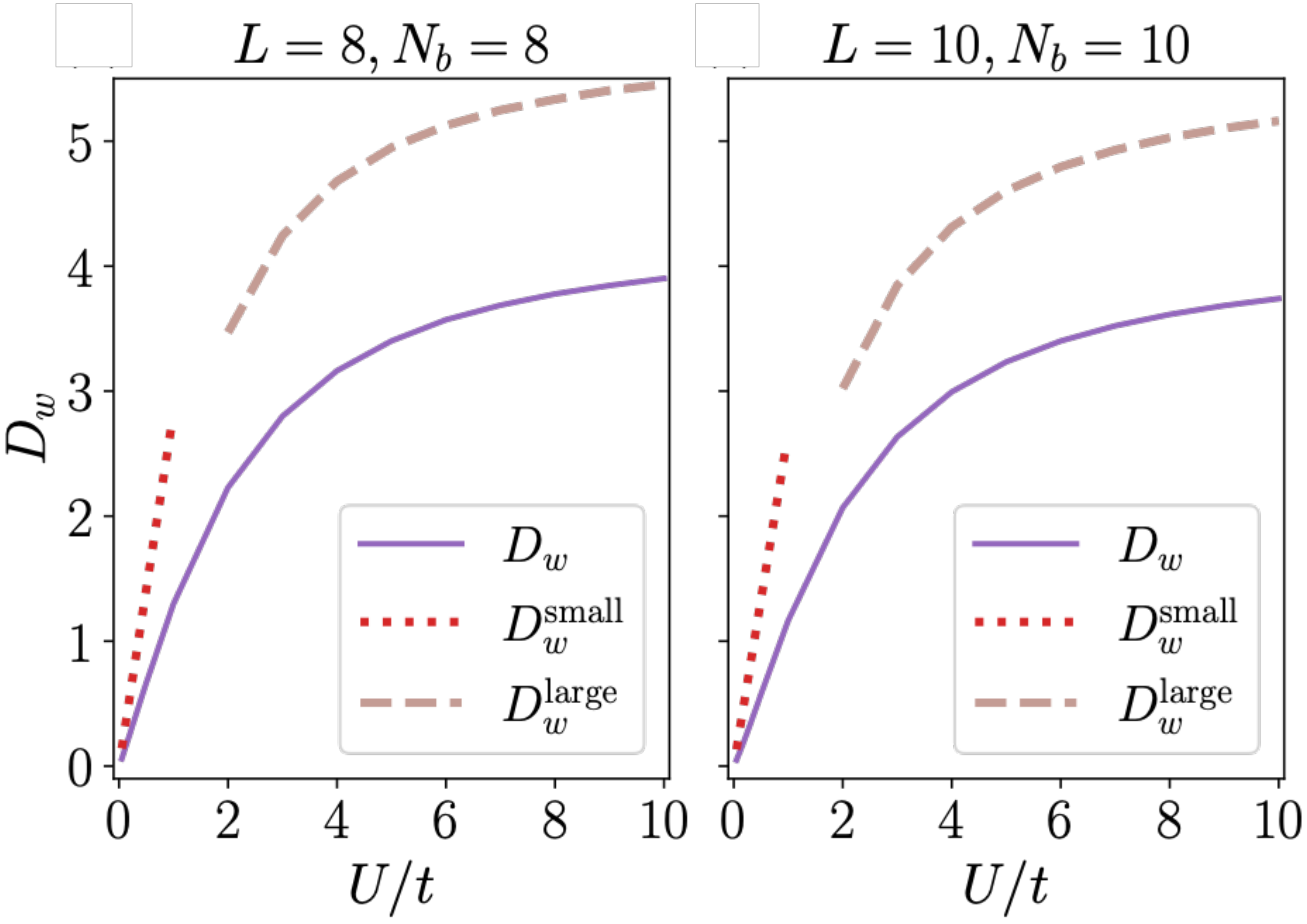
$$\hat{H}(\Phi)^{\mathrm{proj}} = \sum_{j=1}^{L}\left[\frac{U}{4}\hat{W}_j^\dagger\hat{W}_j^\dagger\hat{W}_j\hat{W}_j + \frac{U}{2}\hat{W}_j^\dagger\hat{W}_{j-1}^\dagger\hat{W}_j\hat{W}_{j-1} - \frac{U}{8}\left(e^{-2i\Phi/L}\hat{W}_j^\dagger\hat{W}_j^\dagger\hat{W}_{j-1}\hat{W}_{j-1} + \mathrm{H.c.}\right)\right]$$

Takayoshi et al., *Phys. Rev. A* **88**, 063613 (2013) — Tovmasyan et al., *Phys. Rev. B* **88**, 220510 (2013)

Possible ways to realize the Creutz ladders with ultracold gases include:
Jünemann et al., *Phys. Rev. X* **7**, 031057 (2017) — Kang et al., *New J. Phys.* **22**, 013023 (2020)

The Drude weight for the Creutz ladder

ED results



$$D_w^{\text{small}} < \pi U \left(2n - \frac{2\mathfrak{g}^{\text{proj}}(0)}{L} \right)$$

$$D_w^{\text{large}} < 4\pi t \left(n - \frac{2\mathfrak{g}(0)}{L} \right)$$

The many-body quantum metric reduces the upper bound of the Drude weight.
Large many-body quantum metric means that the eigenstates of the interacting system deviate strongly when a flux is inserted.

Salerno, Ozawa & Törmä, *Phys. Rev. B* **108**, L140503 (2023)

Conclusions

Superfluidity of flat band BECs: quantum distance with the condensate state determines the excitations and the speed of sound — reduces to quantum metric for real eigenstates.

Julku, Salerno & Törmä, *Low Temp. Phys.* **40**, 701 (2023)

Drude weight and many-body quantum metric: exact diagonalization treatment of Bose systems predicts that the many-body version of the quantum metric is a bound for the Drude weight in flat band systems.

Salerno, Ozawa & Törmä, *Phys. Rev. B* **108**, L140503 (2023)

Thanks!

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